Robotics for IEM (WMIE005)

Final Exam, 29 October 2025

Student Number: .....

1. You have **2 hours** for the exam.
2. Before you start, we recommend that you check all the questions and **plan your time** per question.
3. For every question, **indicate the question number** on the answer sheet.
4. Please write down your (i) student ID number, and (ii) seat number, **very clearly** on **all of the pages that you receive**. Please write your answer **using a (preferably blue) pen, not a pencil**.
5. The exam contains four questions with compulsory sub-questions and an optional sub-question for which bonus points can be earned.
6. In order to receive any bonus points, you **must complete the course evaluation survey** and email a screenshot with your student number to the course coordinator.
7. This is a closed-book exam, but you can use a calculator and **the formula sheet** at the end of this document. Please write down your answer clearly and with proper argumentation/reasoning whenever needed. **Providing only the final answer without proper argumentation is not acceptable**.
8. If you are unclear about a specific problem, you can make your own assumptions. Describe your assumptions at the beginning of your answer.
9. Whenever we think it is appropriate, a follow-up oral exam to suspected cases will be arranged before the final grade is determined. In this case, the follow-up oral exam will be based on the questions of this exam and the final grade will be based on the same weighting factor as before where the adjusted grade from the oral exam will be used to replace the final exam grade.
10. If you return the sheets, then your exam will be graded, unless you explicitly write **"do not grade"** on all of the pages. If the exam is graded, it will be registered.

Table 1: This table is for grading purposes only.

	Question 1	Question 2	Question 3	Question 4
Points Obtained	/25	/25	/25	/25
Bonus Points Obtained	/2.5	/2.5	/2.5	/2.5

1. [**25 points**] Answer the following conceptual questions.

   (a) [**8 points**] A robotic manipulator is in an initial state. You are given a set of desired linear and angular velocities of the end-effector frame in the Cartesian space. Describe how the corresponding joint rates can be found. Explain if the corresponding joint rates can always be found. **Hint:** Consider the mathematical relation between the given and the requested variables.

   (b) [**6 points**] Consider the problem of dynamics of a manipulator. (i) Describe the Newton-Euler method by indicating whether it is a force-balance method or an energy-based method and the steps that are involved. (ii) Describe the Lagrangian method by indicating whether it is a force-balance method or an energy-based method and the steps that are involved. **Hint:** you do not need to write down any formulas, rather only mention the physical concepts described by the formulas.

   (c) [**6 points**] Describe the concept of degree of freedom (DOF) required to describe the state of a system. Explain how many degrees of freedom are required to describe the pose of a 3D object in 3D space.

   (d) [**5 points**] Describe the concept of inertia tensor by indicating what it represents. **Hint:** you do not need to write down any formulas, rather only mention the physical concepts described by the formulas.

   (e) [**2.5 points - Bonus**] Describe the necessary conditions for the applicability of Pieper's method in the context of inverse manipulator kinematics.

2. [**25 points**] Consider the PRR manipulator shown in Figure 1.

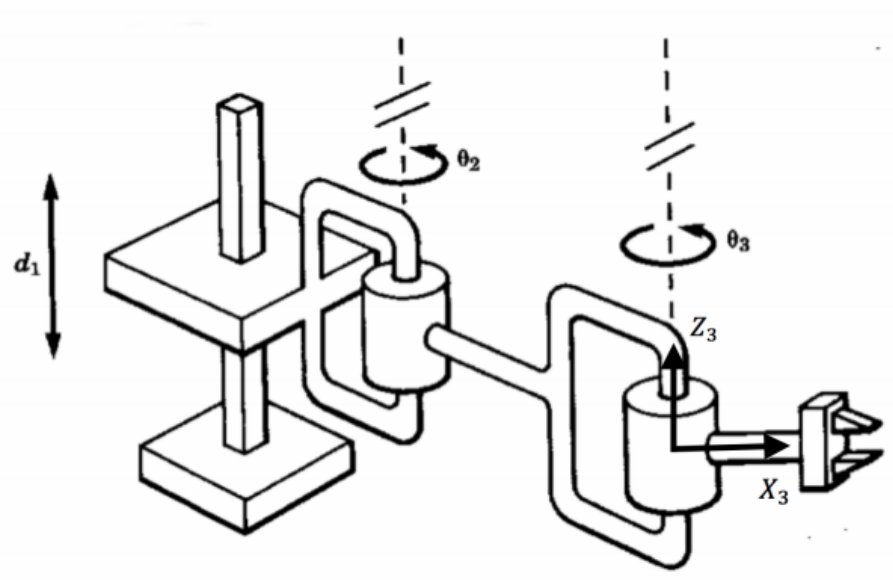


Figure 1: The given PRR manipulator with one prismatic (d_1) and two revolute joints (θ_2 and θ_3). Assume that this image shows a joints configuration where $\theta_2 = \theta_3 = 0$.

(a) [**5 points**] Draw a schematic of this manipulator, with the axes of frames $\{0\}$ through $\{3\}$ labeled. Note that frame $\{3\}$ has been done for you, and your solution should be consistent with the given frame $\{3\}$. Also, include the parameters d_1, a_1, θ_2, a_2 and θ_3 on your schematic. **Hint:** You can assume that Figure 1 shows a configuration where $\theta_2 = \theta_3 = 0$.

(b) [**5 points**] Find the DH parameters for this manipulator. You need to assign the interior frames of this manipulator using the frame assignment convention discussed in class.

Link frame i	α_{i-1}	a_{i-1}	θ_i	d_i
1				
2				
3				

(c) [**5 points**] Derive the forward kinematics, ${}^0_3T$, for this manipulator as a function of ${}^{i-1}_{i}T$. You need to derive ${}^{i-1}_{i}T$ matrices. **Hint:** You do not need to perform the matrix multiplications. See Appendix A.5 for relevant formulas.

(d) [**10 points**] Find the Jacobian matrix (containing both linear and angular parts) in frame 0, namely ${}^0J$, for this manipulator using differentiation of ${}^0\mathbf{P}_{3org}$ if the closed form of the ${}^0_3T$ matrix is given as below. **Hint:** You do not need to perform the matrix multiplications, for the angular part of the Jacobian, you can write the $\hat{Z}_i$ vectors parametrically. See Appendix A.7 for relevant formulas.

$$ {}^0_3T = \begin{bmatrix} c_{23} & -s_{23} & 0 & a_2c_2 + a_1 \\ s_{23} & c_{23} & 0 & a_2s_2 \\ 0 & -c_2 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(e) [**2.5 points - Bonus**] Write the Jacobian matrix in frame 3, namely ${}^3J$, as a function of ${}^0J$. **Hint:** You do not need to perform the matrix multiplications.

3. [**25 points**] Consider the following RRRR manipulator, shown in Figure 2.

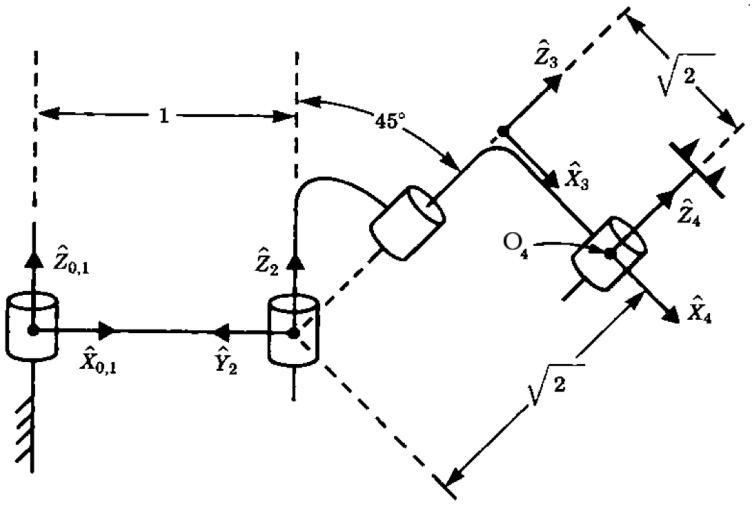


Figure 2: The given RRRR manipulator with four revolute joints (θ_1, θ_2, θ_3, and θ_4). Assume that this image shows a joints configuration where $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$.

(a) [**20 points**] The ${}^{i-1}_{i}T$ matrices are given below. Find the Jacobian matrix (containing both linear J_v and angular J_ω parts) in frame 4, namely ${}^{4}J_\omega$, using the velocity propagation approach. **Hint:** You can write the answer parametrically without simplifying the expressions.

$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} -s_2 & -c_2 & 0 & 1 \\ c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} s_3 & c_3 & 0 & 0 \\ -c_3 c_{\pi/4} & s_3 c_{\pi/4} & -s_{\pi/4} & s_{\pi/4}\sqrt{2} \\ 0 & 0 & c_{\pi/4} & c_{\pi/4}\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} c_4 & -s_4 & 0 & \sqrt{2} \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(b) [**5 points**] If the joints are set to rotate at a fixed rate of $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}_4 = \pi/4$ per second, find the angular and linear velocities of the end-effector frame after $t = 1s$. **Hint:** You can write the answer parametrically by only specifying the joint variable values.

(c) [**2.5 points - Bonus**] Write the linear Jacobian matrix J_v in frame $\{4\}$ for the position $\mathbf{q} = [0, 90°, -90°, 0]^T$, where $\mathbf{q}$ is the vector of joint variables.

4. [**25 points**] Consider the following dynamical system:

$$\begin{cases} \dot{x}_1 = -ax_1 + 4x_2 \\ \dot{x}_2 = -bx_1 - x_2^3 \end{cases} \qquad \text{where} \quad a, b \in (0, +\infty)$$

(a) [**12 points**] Find all the equilibrium points of this system and study their stability by means of the Lyapunov function $V(x_1, x_2) \coloneqq x_1^2 + cx_2^2$, with $c \in R$.

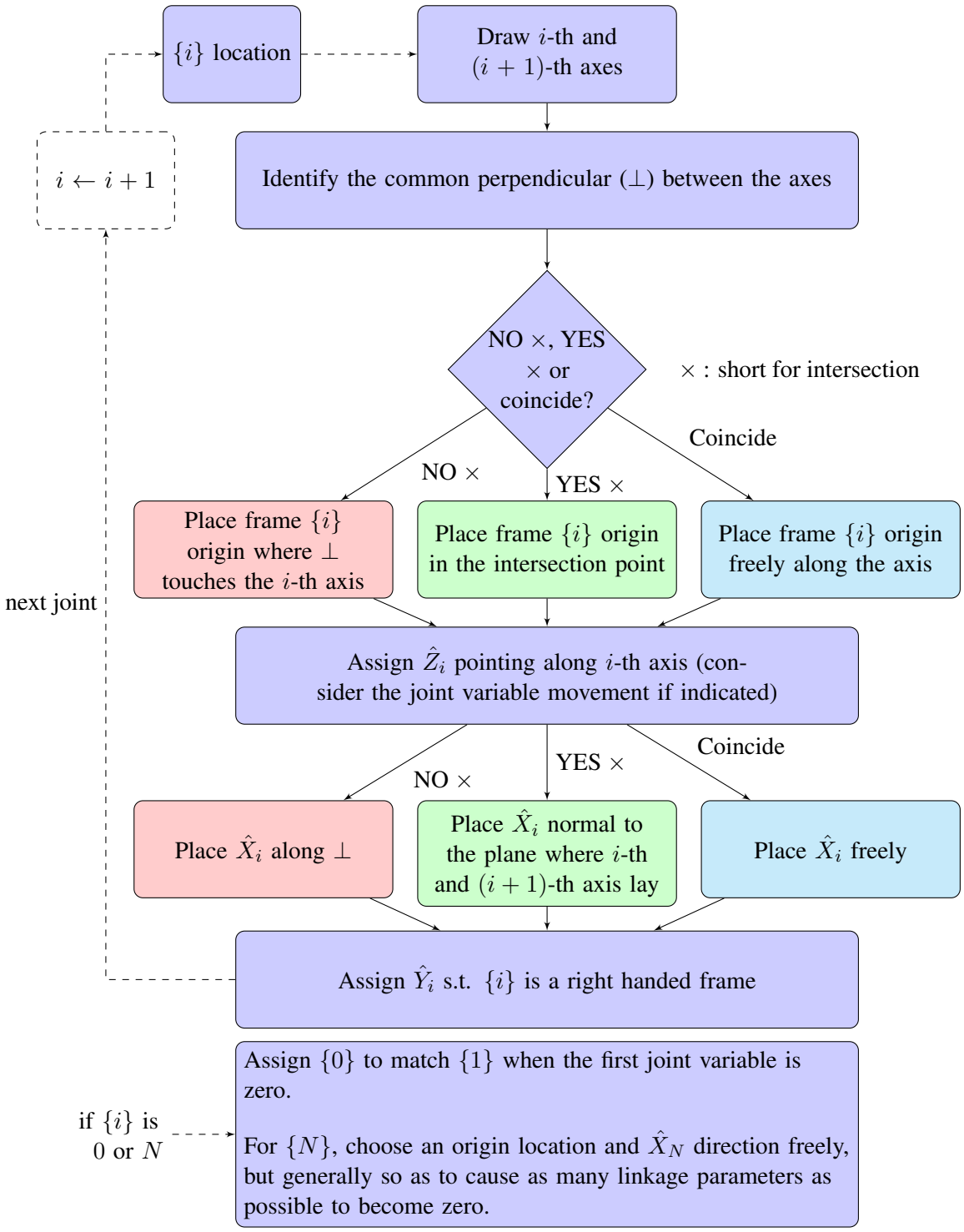
(b) [**8 points**] Describe the concepts of (i) stability and (ii) asymptotic stability in the Lyapanuv sense.

(c) [**5 points**] Consider the system of a **hanging** pendulum and consider two cases (i) the joint does not have any damping and (ii) the joint has damping. The joint does not have springiness in both cases. Without writing the equations of motion, describe the stability property of the system around its equilibrium point. **Hinte:** In your description, consider the **hanging** pendulum to be perturbed from its equilibrium state and discuss time evolution of the perturbation.

(d) [**2.5 point - Bonus**] Consider the system of an inverted pendulum that has a rotational joint with both nonzero damping and torsional spring stiffness. Without writing the equations of motion, discuss qualitatively the stability of the system about its upright equilibrium as the spring stiffness varies.

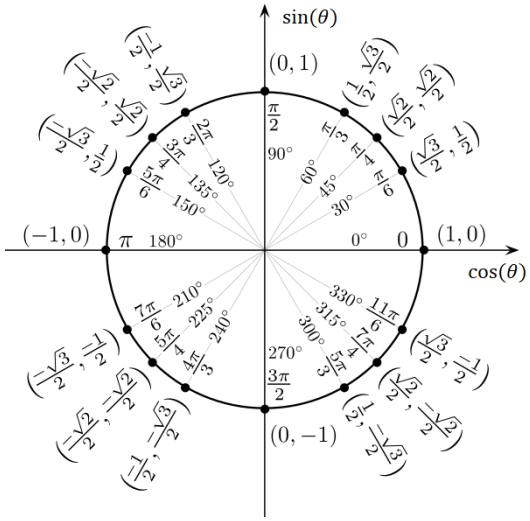
Appendix A - Information Sheet

A.1 Frame assignment



A.2 Trigonometric identities

- $sin(\alpha \pm \beta) = sin(\alpha)cos(\beta) \pm cos(\alpha)sin(\beta)$
- $cos(\alpha \pm \beta) = cos(\alpha)cos(\beta) \mp sin(\alpha)sin(\beta)$
- $sin(-\alpha) = -sin(\alpha)$
- $cos(-\alpha) = cos(\alpha)$
- $\frac{\mathrm{d}}{\mathrm{d}\alpha} sin(\alpha) = cos(\alpha)$
- $\frac{\mathrm{d}}{\mathrm{d}\alpha} cos(\alpha) = -sin(\alpha)$



A.3 Cross product

$$v \times w = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

A.4 Right-hand coordinate system

$\hat{x} = \hat{y} \times \hat{z} \qquad \hat{y} = \hat{z} \times \hat{x} \qquad \hat{z} = \hat{x} \times \hat{y}$

A.5 Basic rotation matrices

$$R_z(\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

A.6 General link transformation matrix

$$^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A.7 Angular and linear velocities propagation

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right) + \dot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$

A.8 Jacobian matrix

Suppose that the forward kinematics of a robotic manipulator of two degrees of freedom is given by ${}^0P = f(q)$, where $q = [q_1, q_2]^\top$ are the joints variables. Thus, the linear velocities v and angular velocities ω in the workspace and $\dot{q}$ in the joint space are related by the following formulas, where ϵ_i is 0 for prismatic and 1 for revolute joint i.

$$ {}^0v = {}^0J_v(q)\dot{q} \iff {}^0v = \begin{bmatrix} \partial f_1/\partial q_1 & \partial f_1/\partial q_2 \\ \partial f_2/\partial q_1 & \partial f_2/\partial q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} $$

$$ {}^0\omega = {}^0J_\omega(q)\dot{q} \iff {}^0\omega = \begin{bmatrix} \epsilon_1 {}^0\hat{Z}_1 & \epsilon_2 {}^0\hat{Z}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} $$

A.9 Kinetic and potential energy

The total kinetic energy of a rigid robotic system with n links is given by

$$ T(q, \dot{q}) = \sum_{i=1}^{n} \left(\frac{1}{2} m_i v_{ci}^\top v_{ci} + \frac{1}{2} \omega_i^\top I_{ci} \omega_i \right) = \frac{1}{2} \dot{q}^\top M(q) \dot{q} $$

where $m_i, v_{ci}, \omega_i, I_{ci}$ are the mass, linear velocity, angular velocity and inertia (at the center of mass frame $\{c_i\}$) of the i-th link, respectively. The matrix $M(q)$ is the inertia matrix which is symmetric and positive definite. The potential energy is given by

$$ U(q) = \sum_{i=1}^{n} u_i, \text{ where } u_i = -m_i {}^0g^\top {}^0P_{ci} + u_{ref_i} $$

with ${}^0g$ the 3×1 gravity vector, ${}^0P_{ci}$ is the vector locating the center of mass of the i-th link, and u_{ref_i} is a constant chosen so the minimum value of u_i is zero.

A.10 Newton-Euler Iterative approach

Outward iterations:

$$ \begin{aligned} {}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\, {}^i\omega_i + \dot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} \\ {}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}_{i}R\, {}^i\dot{\omega}_i + {}^{i+1}_{i}R\, {}^i\omega_i \times \dot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} + \ddot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} \\ {}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R \left({}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times \left({}^i\omega_i \times {}^iP_{i+1} \right) + {}^i\dot{v}_i \right) \\ {}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}} \right) + {}^{i+1}\dot{v}_{i+1} \end{aligned} $$

Net forces and torques:

$$ \begin{aligned} {}^{i+1}F_{i+1} &= m_{i+1} {}^{i+1}\dot{v}_{C_{i+1}} \\ {}^{i+1}N_{i+1} &= {}^{C_{i+1}}I_{i+1} {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1} {}^{i+1}\omega_{i+1} \end{aligned} $$

Inward iterations:

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R^{i+1}f_{i+1} + {}^{i}F_i \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R^{i+1}f_{i+1}
\end{aligned}
$$

Joint torques:

$$\tau_i = {}^{i}n_i^T \, {}^{i}\widehat{Z}_i$$

A.11 Lagrangian approach

Consider the Lagrangian function

$$L(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

The equations of motion for a robotic manipulator are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial K}{\partial q}(q, \dot{q}) + \frac{\partial U}{\partial q}(q) = \tau$$

where τ is the vector of joint torques.

A.12 Inertia tensor

The inertia tensor with respect to frame A is calculated as the, where ρ is the density of the object:

$${}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

- $I_{xx} = \iiint (y^2 + z^2)\rho \, dx \, dy \, dz$
- $I_{yy} = \iiint (x^2 + z^2)\rho \, dx \, dy \, dz$
- $I_{zz} = \iiint (x^2 + y^2)\rho \, dx \, dy \, dz$
- $I_{xy} = \iiint xy\rho \, dx \, dy \, dz$
- $I_{xz} = \iiint xz\rho \, dx \, dy \, dz$
- $I_{yz} = \iiint yz\rho \, dx \, dy \, dz$

A.13 Control law partitioning for manipulator control

For the problem of manipulator control, we have the equations of motion as:
$\tau = M(q)\ddot{q} + V(q, \dot{q}) + G(q) + F(q, \dot{q})$
The model-based portion of the control law can be written as:
$\tau = \alpha\tau' + \beta$

$\alpha = M(q)$

$\beta = V(q, \dot{q}) + G(q) + F(q, \dot{q})$
The servo-based portion of the control law can be written as:
$\tau' = \ddot{q}_d + k_v\dot{E} + k_pE$

$E = q - q_d$

A.14 Law of Cosines

- $cos(\alpha) = \frac{b^2+c^2-a^2}{2bc}$
- $cos(\beta) = \frac{a^2+c^2-b^2}{2ac}$
- $cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$

